

CAUSAL PROPAGATORS FOR ALGEBRAIC GAUGES

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Abstract

Applying the principle of analytic extension for generalized functions we derive causal propagators for algebraic non-covariant gauges. The so generated manifestly causal gluon propagator in the light-cone gauge is used to evaluate two one-loop Feynman integrals which appear in the computation of the three-gluon vertex correction. The result is in agreement with that obtained through the usual prescriptions.

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Over a quarter of a century ago Bollini, Giambiagi, and Domínguez^[1,2] considered causal distributions in the context of the Fourier transform of radial functions, $f(Q^2)$, where $Q^2 \equiv k_0^2 - \vec{k}^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$. By introducing a positive parameter α such that

$$(f_\alpha, \tau) = (f(\alpha^2 k_0^2 - \vec{k}^2), \tau(k_0, k_1, k_2, k_3)) \equiv \frac{1}{\alpha} (f(Q^2), \tau(\frac{k_0}{\alpha}, k_1, k_2, k_3)), \quad (1)$$

where τ is a test function, one defines that f_α is analytic in α if for any τ , the functional (f_α, τ) is also analytic in α .

Now, when f_α is analytically continued to the whole of the upper half plane of α , then a causal distribution is defined through the following extension $k_0 \rightarrow \alpha k_0$, i.e.,

$$f(k^2 + i\epsilon) = \lim_{\alpha \rightarrow 1+i\epsilon} f(\alpha^2 k_0^2 - \vec{k}^2), \quad \epsilon \rightarrow 0^+. \quad (2)$$

From this reasoning of analytic continuation as a postulate, one can derive the covariant Feynman propagator in momentum space as follows

$$\frac{1}{k^2} \rightarrow \lim_{\alpha \rightarrow 1+i\epsilon} \frac{1}{\alpha^2 k_0^2 - \vec{k}^2} = \frac{1}{k^2 + 2i\epsilon k_0^2}, \quad \epsilon \rightarrow 0^+. \quad (3)$$

Since $k_0^2 > 0$, one has the usual prescription for handling covariant poles, namely,

$$\frac{1}{k^2} \rightarrow \lim_{\varepsilon \rightarrow 0^+} \frac{1}{k^2 + i\varepsilon}, \quad \varepsilon \equiv 2\epsilon k_0^2 \rightarrow 0^+. \quad (4)$$

Non-covariant (or algebraic) gauge choices which are characterized by an external, constant vector n_μ , on the other hand leads to the appearance of gauge-dependent poles¹ $(k \cdot n)^{-\alpha}$, $\alpha = 1, 2, \dots$ in the Feynman amplitudes. These therefore contain in their structure, for instance, factors of the form

$$\frac{1}{(k^2 + i\varepsilon)[k \cdot n]} \quad ,$$

where

$$\frac{1}{[k \cdot n]}$$

¹very often, incorrectly called “unphysical” or “spurious” poles.

indicates that one still needs a mathematically consistent way of dealing with it at the pole $k \cdot n = 0$. Several prescriptions have been defined and used in the literature for this purpose. However, mathematics only does not suffice for such a task as it has been demonstrated in the particular case of the light-cone choice^[3]; we also need to watch out that causality is not violated by the prescription *per se* or even in the process of its implementation in a direct calculation. This is the reason why one should not consider gauge-dependent poles as “unphysical”, since they too must be constrained by causality, so that in our field theory we will not allow that positive-energy quanta propagating into the future become mixed up with negative-energy ones propagating into the past and vice-versa.

With the insight acquired in the work of reference [3], recently Pimentel and Suzuki^[4] have proposed a causal prescription for the light-cone gauge starting from the premise that the propagator as a whole must be causal. In this sequel, we propose that within the framework of analytic continuation as discussed above, we can arrive at the very causal prescription for the light-cone gauge. One notes, however, that non-covariant poles (such as the light-cone one) are not of a radial type function and no proof is given that such are tempered distributions either. Yet, on the assumption that analytic extension as defined above for the covariant pole is legitimate and applicable to non-covariant poles, we draw some interesting results.

To begin with, consider the product $(k^2 k \cdot n)^{-1}$ with $n^\mu \equiv (n^0, 0, 0, n^3)^2$ being an external, arbitrary vector which determines the choice of a gauge of the algebraic or non-covariant type. The factor $(k^2 k \cdot n)^{-1}$ upon the hypothesis of analytic continuation becomes

$$\frac{1}{k^2 k \cdot n} \rightarrow \frac{1}{(k^2 + 2i\epsilon k_0^2)(k \cdot n + i\epsilon k^0 n^0)}. \quad (5)$$

As long as the external vector n is quite arbitrary, we can choose it so that $n^0 > 0$, and since ϵ is strictly positive, equation (5) may be rewritten as³

$$\begin{aligned} \frac{1}{k^2 k \cdot n} &\rightarrow \frac{1}{(k^2 + 2i\epsilon k_0^2)(k \cdot n + i\epsilon |k^0| n^0)}, & \text{for } k^0 > 0 \\ \frac{1}{k^2 k \cdot n} &\rightarrow \frac{1}{(k^2 + 2i\epsilon k_0^2)(k \cdot n - i\epsilon |k^0| n^0)}, & \text{for } k^0 < 0 \end{aligned} \quad (6)$$

²for convenience we have chosen components $n^1 = n^2 = 0$

³recall that we continue analytically to the whole of the *upper* half plane of α

or, using the Heaviside Θ -function,

$$\frac{1}{k^2 k \cdot n} \rightarrow \frac{1}{k^2 + i\varepsilon} \left\{ \frac{\Theta(-k^0)}{k \cdot n - i\xi} + \frac{\Theta(k^0)}{k \cdot n + i\xi} \right\}, \quad \begin{aligned} \varepsilon &\equiv 2\epsilon k_0^2 \rightarrow 0^+ \\ \xi &\equiv \epsilon |k^0| n^0 \rightarrow 0^+ \end{aligned}, \quad (7)$$

which is exactly the causal prescription considered in reference [4].

We can, of course, generalize for higher order poles of $(k \cdot n)$ as well as for non-covariant gauge choices other than the light-cone one. We shall only consider the double pole case and briefly discuss the pure homogeneous axial gauge ($n^0 = 0$) and the pure homogeneous temporal gauge ($n^3 = 0$) choices for n^μ .

First of all, let us consider the simple pole cases. From equation (5) we note that the analytic continuation of $k \cdot n$ entails a sign dependence of the imaginary part coming from k^0 and n^0 .

The pure temporal case⁴ is such that there will be a violation of causality if one employs the principal-value (PV) prescription to treat the pole $(k \cdot n)^{-1}$, in the same manner as it breaks causality in the light-cone case^[3]. Indeed, evaluation of the Wilson loop to the fourth order carried out by Caracciolo *et al*^[5] has shown that in the temporal gauge the PV prescription used to treat the gauge-dependent poles leads to results which do not agree⁵ with the ones obtained in the Feynman and Coulomb gauges.

On the other hand, one has a very different situation for the pure axial gauge for which $n^\mu = (0, 0, 0, n^3)$. Here, nothing whatsoever can be said *a priori* whether a given prescription to treat the gauge dependent pole $(k \cdot n)^{-1}$ will or will not violate causality, since analytic continuation for this peculiar case is ill-defined.

Secondly, let us consider the double pole $(k \cdot n)^{-2}$ cases. In the light-cone gauge, the appearance or no of the double pole factor $(k \cdot n)^{-2}$ in the Feynman amplitudes depends upon whether one uses the four-component uneliminated formalism or the eliminated two-component formalism. In the latter case, it arises, for instance, in the evaluation of the one-loop gluon self-energy. A sample calculation of a typical integral of this type has been presented in reference [3], where use of the causal prescription (or any other prescription which preserves causality) is *mandatory*.

⁴we stick to the case $n^0 > 0$

⁵not surprisingly since causality has been broken by the PV prescription

Finally, a word on the axial gauge when one chooses for the external vector $n^\mu = (n^0, 0, 0, n^3)$ such that $(n^0)^2 < (n^3)^2$. In this case, one *has to* use the causal prescription (or any other prescription preserving causality) in order to circumvent the gauge dependent poles.

We now proceed by implementing this causal vector boson propagator in two types of integrals which occur in the evaluation of the “swordfish” diagrams of the three-gluon vertex correction when we employ the two-component formalism of the light-cone gauge^[7]. We shall see that the outcome is concordant with the result obtained through the use of the other prescriptions^[6]. In order to do this we follow previous notation and conventions as employed in reference [7]. The integrals are:

$$\mathcal{K}(p, q) = \int \frac{d^2\omega r}{r^2(r-q)^2} \frac{1}{(p^+ + r^+)} \quad (8)$$

and

$$\mathcal{K}^l(p, q) = \int \frac{d^2\omega r}{r^2(r-q)^2} \frac{(p^l + r^l)}{(p^+ + r^+)} \quad , \quad l = 1, 2, \quad (9)$$

which can be rewritten in a more convenient way as

$$\tilde{\mathcal{K}}(p, q) = \int \frac{d^2\omega r}{(r-p)^2(r-p-q)^2} \frac{1}{r^+} \quad , \quad (10)$$

and

$$\tilde{\mathcal{K}}^l(p, q) = \int \frac{d^2\omega r}{(r-p)^2(r-p-q)^2} \frac{r^l}{r^+} \quad . \quad (11)$$

The singularities at $r^+ = 0$ are treated according to what the principle of causality obliges for the whole of the boson propagator, namely,

$$\frac{1}{r^2 r^+} \rightarrow \frac{1}{r^2 + i\varepsilon} \left[\frac{\theta(r^0)}{r^+ + i\varepsilon} + \frac{\theta(-r^0)}{r^+ - i\varepsilon} \right] \quad , \quad (12)$$

where the infinitesimals ε and ϵ goes to zero from above, i.e., they are small, positive real numbers, and $\theta(\pm x)$ is the usual Heaviside unit step function. This propagator naturally ensures that positive-energy quanta propagating into the future do not become mixed up with negative-energy ones, and vice-versa into the past.

For the actual computation we decompose the momentum integration into its longitudinal and transverse parts, so that for the longitudinal part

we can regularize the integral via dimensional regularization in an Euclidean space of $2\omega - 2$ dimensions and for the transverse part we use

$$\frac{\theta(-r^0)}{r^+ - i\epsilon} + \frac{\theta(r^0)}{r^+ + i\epsilon} = PV \frac{1}{r^+} - i\pi\delta(r^+) \frac{(r^+ + r^-)}{|r^+ + r^-|} , \quad (13)$$

with

$$PV \frac{1}{r^+} = \frac{1}{2} \left[\frac{1}{r^+ + i\epsilon} + \frac{1}{r^+ - i\epsilon} \right] , \quad (14)$$

After some algebra, we arrive at the following partial results

$$\tilde{\mathcal{K}}(p, q) = i \frac{(-\pi)^\omega \Gamma(2 - \omega)}{(p^+ + q^+)} \left\{ (q^2)^{\omega-2} \int_0^1 dy \mathcal{F}(y) - (\hat{q}^2)^{\omega-2} \int_0^1 dy \mathcal{G}(y) \right\} , \quad (15)$$

and

$$\begin{aligned} \tilde{\mathcal{K}}^l(p, q) &= (p^l + q^l) \tilde{\mathcal{K}}(p, q) - i(-\pi)^\omega \Gamma(2 - \omega) \frac{q^l}{p^+ + q^+} \\ &\times \left\{ (q^2)^{\omega-2} \int_0^1 dy y \mathcal{F}(y) - (\hat{q}^2)^{\omega-2} \int_0^1 dy y \mathcal{G}(y) \right\} , \end{aligned} \quad (16)$$

where

$$\mathcal{F}(y) \equiv \frac{[y(1 - y)]^{\omega-2}}{(1 - \sigma y)} , \quad (17)$$

$$\mathcal{G}(y) \equiv \frac{[(y - \xi)(y - \bar{\xi})]^{\omega-2}}{(1 - \sigma y)} , \quad (18)$$

$$\sigma \equiv \frac{q^+}{(p^+ + q^+)} , \quad (19)$$

$$\xi \equiv \frac{(1 + \nu - \rho) + \sqrt{(1 + \nu - \rho)^2 - 4\nu}}{2} , \quad (20)$$

$$\bar{\xi} \equiv \frac{(1 + \nu - \rho) - \sqrt{(1 + \nu - \rho)^2 - 4\nu}}{2} , \quad (21)$$

$$\nu \equiv \frac{2(p^+ + q^+)(p^- + q^-)}{\hat{q}^2} , \quad (22)$$

$$\rho \equiv \frac{2p^+ p^-}{\hat{q}^2} , \quad (23)$$

$$\hat{q}^2 \equiv q^1 q^1 + q^2 q^2 = 2q^+ q^- - q^2 , \quad (24)$$

$$p^\pm \equiv \frac{(p^0 \pm p^3)}{\sqrt{2}} . \quad (25)$$

These results, Eqs. (15) and (16), agree with those of reference [7]. Moreover, in what follows we are going to explore a little different pathway to evaluate de y -integrals. We first take the limit $\omega \rightarrow 2$ and then perform the y -integrations. Doing this enables us to obtain a “better looking” form for the final results.

The end results for the integrals in question are:

$$\tilde{\mathcal{K}}(p, q) = \frac{i\pi^2}{q^+} T(p, q) + \mathcal{O}(2 - \omega) , \quad (26)$$

where

$$\begin{aligned} T(p, q) \equiv & \ln(1 - \sigma) \ln \left(\frac{q^2}{\nu \hat{q}^2} \right) + \ln \left(\frac{\xi}{\xi - 1} \right) \ln \left(\frac{\sigma - 1}{\sigma \xi - 1} \right) \\ & + \ln \left(\frac{\bar{\xi}}{\bar{\xi} - 1} \right) \ln \left(\frac{\sigma - 1}{\sigma \bar{\xi} - 1} \right) + \mathcal{S}(\sigma) \\ & - \mathcal{S} \left(\frac{\sigma}{\sigma - 1} \right) + \mathcal{S} \left(\frac{\sigma \xi}{\sigma \xi - 1} \right) + \mathcal{S} \left(\frac{\sigma \bar{\xi}}{\sigma \bar{\xi} - 1} \right) \\ & - \mathcal{S} \left(\frac{\sigma(\xi - 1)}{\sigma \xi - 1} \right) - \mathcal{S} \left(\frac{\sigma(\bar{\xi} - 1)}{\sigma \bar{\xi} - 1} \right) , \end{aligned} \quad (27)$$

and

$$\tilde{\mathcal{K}}^l(p, q) = \frac{(p^l q^+ - p^+ q^l)}{q^+} \tilde{\mathcal{K}}(p, q) - i\pi^2 \frac{q^l}{q^+} U(p, q) + \mathcal{O}(2 - \omega) , \quad (28)$$

where

$$U(p, q) \equiv \ln \left(\frac{q^2}{\nu \hat{q}^2} \right) - (\xi - 1) \ln \left(\frac{\xi}{\xi - 1} \right) - (\bar{\xi} - 1) \ln \left(\frac{\bar{\xi}}{\bar{\xi} - 1} \right) . \quad (29)$$

In conclusion, we would like to emphasize and observe that we were able to express Eqs. (26) and (28) in terms of products of logarithms and in terms of various dilogarithms or Spence integrals, $\mathcal{S}(\lambda)$, λ being a general argument

for the dilogarithm. Although much more complex than the basic one-loop light-cone integral, namely,

$$\begin{aligned}\mathcal{K}(p) &\equiv \int \frac{d^{2\omega} r}{r^2(r-p)^2 r^+} \\ &= \frac{i\pi^2}{p^+} \left\{ \frac{\pi^2}{6} - \mathcal{S}(\lambda) \right\} + \mathcal{O}(2-\omega) \ ,\end{aligned}\tag{30}$$

where, in this last equation, λ stands for $\frac{-\hat{p}^2}{p^2}$, Eqs. (26), (28), and (30) show us that basically they belong to the same class of one-loop finite light-cone integrals. The naive power counting to assess the degree of divergence of these integrals remains valid.

REFERENCES

- [1] Bollini, C.G., Giambiagi, J.J., and Domínguez, A.González *Journal of Math. Phys.* **6**, 165 (1965)
- [2] Bollini, C.G., and Giambiagi, J.J. *Il Nuovo Cimento* **34**, 1146 (1965)
- [3] Pimentel, B.M., and Suzuki, A.T. *Phys. Rev.* **D42**, 2115 (1990)
- [4] Pimentel, B.M., and Suzuki, A.T. *Mod. Phys.Lett. A* **6**, 2649 (1991)
- [5] S.Caracciolo, G.Curci, and P.Menotti, *Phys.Lett.* **113B**, 311 (1982)
- [6] S.Mandelstam, S., *Nucl. Phys.* **B 123**, 149 (1983); G.Leibbrandt, *Phys. Rev.* **D 29**, 1699 (1984)
- [7] Suzuki, A. T., *J. Math. Phys.* **29** 1032 (1988)